Little's Law, Operational Laws

University of Tehran
Prepared by: Dr ahmad Khonsari
The University of Tehran
Excerpt from : **Performance Modeling and Design of Computer Systems**
Queueing Theory in Action
Mor Harchol-Balter

Little's Law, Operational Laws

- **Theorem 6.1 (Little's Law for Open Systems)** *For any ergodic open system we have that*
- $\mathbf{E}[N] = \lambda \mathbf{E}[T]$

where $\mathbf{E}[N]$ *is the expected number of jobs in the system,* λ *is the average arrival rate into the system, and* $\mathbf{E}[T]$ *is the mean time jobs spend in the system.*

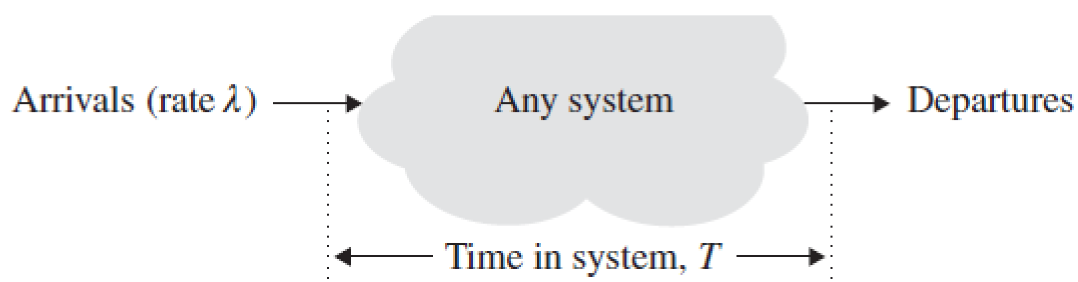


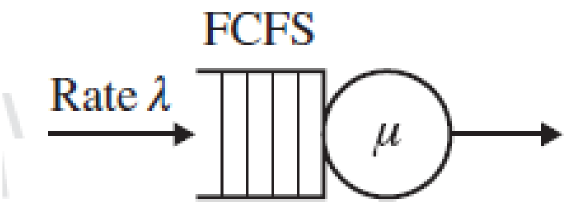
Figure 1. Setup for Little's Law

Little’s Law, Operational Laws

- Note: that Little’s Law makes no assumptions about the arrival process, the service time distributions at the servers, the network topology, the service order, or anything!
- In studying Markov chains, we see many techniques for computing $\mathbf{E}[N]$.
- Applying Little’s Law will then immediately yield $\mathbf{E}[T]$.

Little's Law, Operational Laws

- Consider a single FCFS queue, shown in Fig 2.
- A customer arrives and sees **E**[N] jobs in the system.
- The expected time for each customer to complete is $1/\lambda$ (not $1/\mu$), since the average rate of completions is λ. Hence the expected time until the customer leaves is **E**[T] ≈ $(1/\lambda)$.**E**[N]
- **Fig2.** Little's Law applied to a single server



Little's Law, Operational Laws

- **Theorem 2 (Little's Law for Closed Systems)** *Given any ergodic closed system,*

$N = X \cdot \mathbf{E}[T]$,

where N is a constant equal to the multiprogramming level, X is the throughput (i.e., the rate of completions for the system), and $\mathbf{E}[T]$ *is the mean time jobs spend in the system.*

Little's Law, Operational Laws

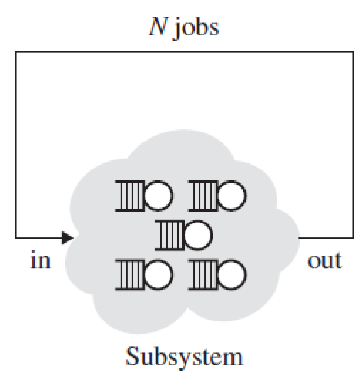
- Fig3 shows a batch system and an interactive (terminal-driven) system.
- Note that for the interactive system (right), the time in system, *T*, is the time to go from "out" to "out," whereas response time, *R*, is the time from "in" to "out."
- Specifically, for a *closed interactive system*, we define

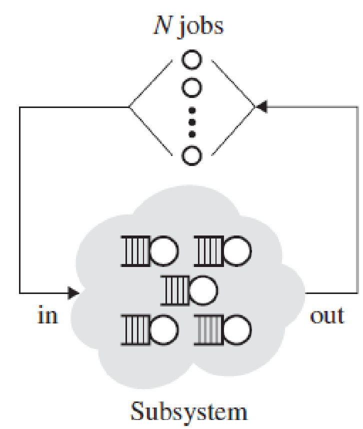
E[*T*] = **E**[*R*] + **E**[*Z*]

where **E**[*Z*] is the average think time,

E[*T*] is the average time in system,

E[*R*] is the average response time.

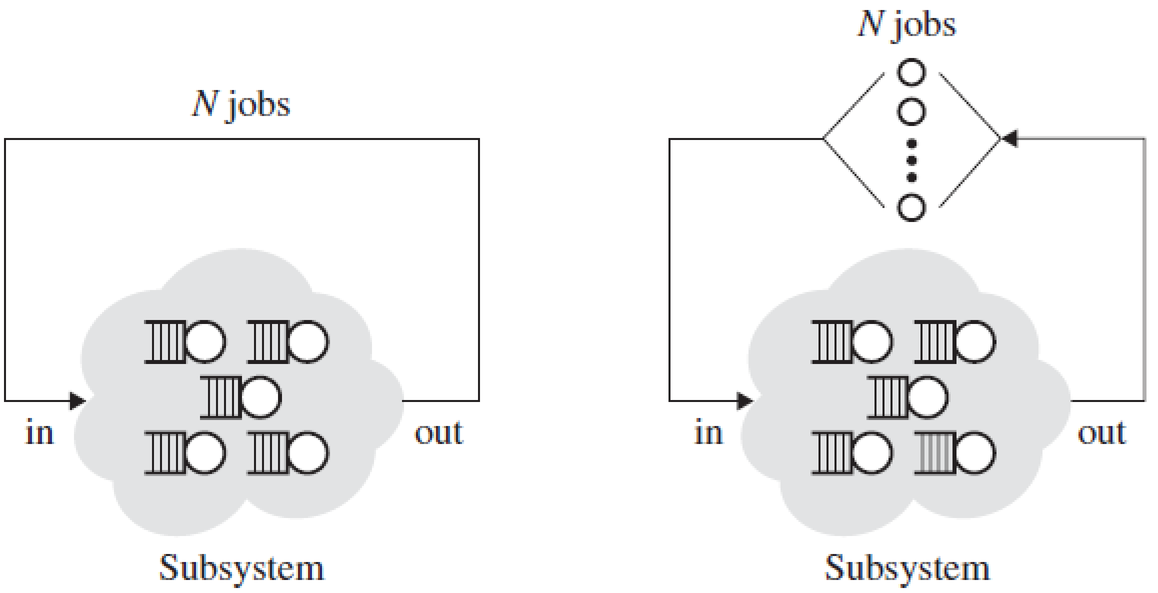




Little's Law, Operational Laws

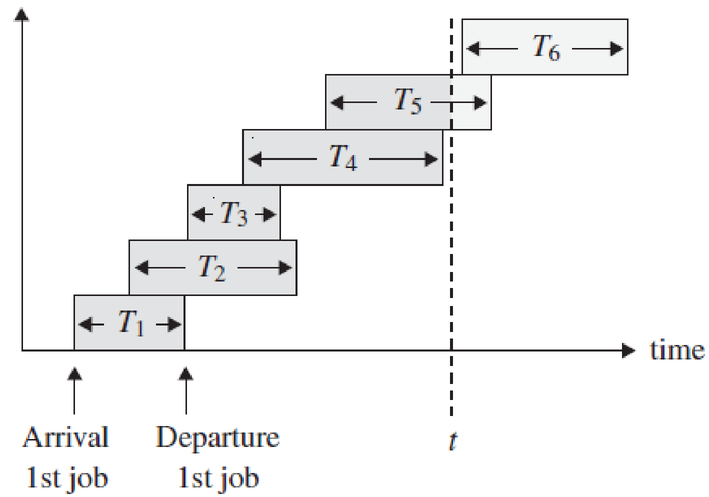
- Note: for open systems and closed batch systems, we refer to **E**[T] as mean response time,
- whereas for closed interactive systems **E**[T] represents the mean time in system and **E**[R] is the mean response time, since response time does not include thinking.
- for an *open system*, throughput and mean response time are uncorrelated.
- By contrast, Little's Law tells us that, for a *closed* system, X and **E**[T] are inversely related, as are X and **E**[R].
- ***Thus in a closed system, improving response time results in improved throughput and vice versa.***

Little's Law, Operational Laws



- **Fig3.** Closed systems: A batch system (left) and an interactive system (right).

Little’s Law for Open Systems



- **Q:** Does this argument

depend on service order?

A: No. Observe that the second

arrival departs *after* the third arrival departs.

- **Q:** Does this argument depend on number of servers?

A: No, this argument holds for any system.

- You may apply the little’s law to any parts of the system including the **server** and **queue** itself.

Little's Law for Open Systems-Queue

- **Corollary 4 (Little's Law for Time in Queue)** *Given any system where $\overline{N}_Q^{\text{Time Avg}}$, $\overline{T}_Q^{\text{Time Avg}}$, λ, and X exist and where λ = X, then*

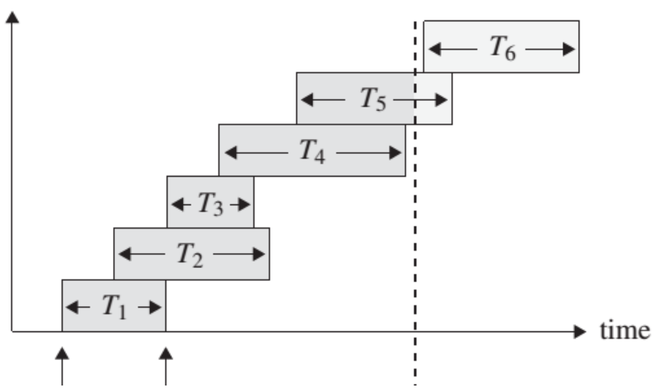
$\overline{N}_Q^{\text{Time Avg}} = \lambda . \overline{T}_Q^{\text{Time Avg}}$

where N_Q represents the number of jobs in queue in the system and T_Q represents the time jobs spend in queues.

- $proof$: Same proof as for Theorem 3, except that now instead of drawing T_i, we draw $T_Q(i)$, i.e. the time of the ith arrival in queues (wasted time).

- Note that $T_Q(i)$ may not be a solid rectangle.

It may be made up of **several rectangles** because the ith job might be in queue for a while, then in service, then waiting in some other queue, then in service, again, etc.



Little's Law for Open Systems- Utilization Law

- **Corollary 5 (Utilization Law)** *Consider a single device i with average arrival rate λ_i jobs/sec and average service rate μ_i jobs/sec, where $\lambda_i < \mu_i$.*
- *Let ρ_i denote the long-run fraction of time that the device is busy.*
- *Then* $\rho_i = \dfrac{\text{Average service time required by a job}}{\text{Average time between arrivals}}$

$$= \frac{1/\mu_i}{1/\lambda_i} = \frac{\lambda_i}{\mu_i}$$

- We refer to ρ_i as the "device utilization" or "device load."
- Observe that, given ergodicity, ρ_i represents both the long-run fraction of time (time average) that device *i* is busy and also the limiting probability (ensemble average) that device *i* is busy.

Little's Law for Open Systems- Utilization Law

- ***Proof:*** Let the "system" consist of the server, as shown in the shaded box of Figure 6. Now the number of jobs in the "system" is always just 0 or 1.

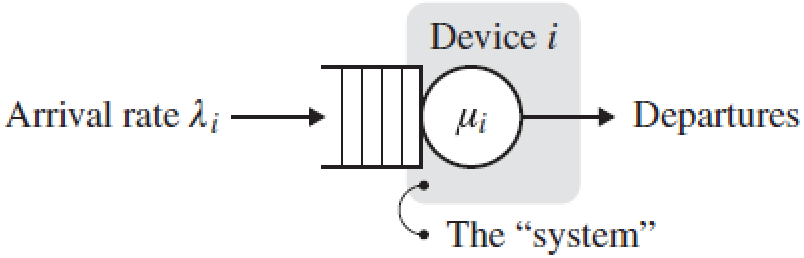


Figure 6. Using Little's Law to prove the Utilization Law

Little's Law for Open Systems-server

- **Question:** What is the expected number of jobs in the system (server) as we have defined it?
- **Answer:** The number of jobs in the system is 1 when the device is busy (this happens with probability ρ_i) and is 0 when the device is idle (with probability $1 - \rho_i$).
- Hence the ***expected* number** of jobs in the **server=** ((1x ρ_i +0x($1 - \rho_i$))) **=** $\boldsymbol{\rho_i}$.
- So, applying Little's Law, we have

ρ_i = Expected number jobs in service facility for device *i*

= (Arrival rate into service facility) · (Mean time in service facility)

= λ_i · **E**[Service time at device *i*] = $\lambda_i \cdot \dfrac{1}{\mu_i}$ ▪

Little's Law for Open Systems-server

- We often express the Utilization Law as

$\rho_i = \lambda_i \mathbf{E}[S_i] = X_i \mathbf{E}[S_i]$

- where ρ_i , λ_i , X_i , and $\mathbf{E}[S_i]$ are the load, average arrival rate, average throughput, and average service requirement at device i, respectively.
- **Question:** Suppose we are only interested in "red" jobs, where "red" denotes some type of jobs. Can we apply Little's Law to just "red" jobs? Prove it.
- **Answer:** Yes.
- **E**[Number of red jobs in system] = $\lambda_{\text{red}} \cdot$ **E**[Time spent in system by red jobs]
- The proof is exactly the same as before, but only the T_i's corresponding to the red jobs are included in Figure 5.

Appendix

- **Appendix:**

Proof of Little's Law for Open Systems

Proof of Little's Law for Open Systems

Let

$A(t)$ = the number of arrivals by time t

$C(t)$ = the number of system completions (departures) by time t.

- Little's Law is actually stated as a relationship between time averages

Let $\lambda = \lim_{t\to\infty} \frac{A(t)}{t}$ and $X = \lim_{t\to\infty} \frac{C(t)}{t}$

- it is typically the case that $\lambda = X$ (one could have $\lambda > X$ if some arrivals get dropped, or if some jobs get stuck and never complete for some reason).

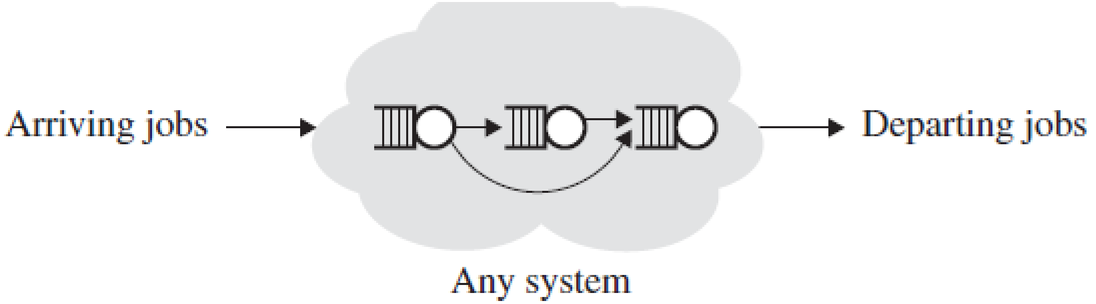


Figure 4. Open system.

Proof of Little's Law for Open Systems

- **Theorem 3 (Little's Law for Open Systems Restated)** *Given any system where* $\overline{N}^{\text{Time Avg}}$*,* $\overline{T}^{\text{Time Avg}}$*, λ, and X exist and where λ = X, then*

$\overline{N}^{\text{Time Avg}} = \lambda . \overline{T}^{\text{Time Avg}}$

- It is an equality between *time averages*, not *ensemble averages*.(i.e. the time-average number in system for sample path *ω* is equal to *λ* times the time-average time in system for that sample path.)

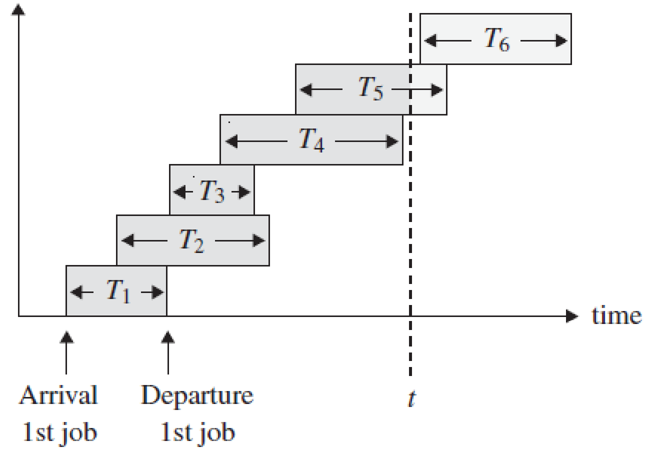
Proof of Little's Law for Open Systems

- For *ergodic*, the time average converges to the ensemble average with prob. 1; i.e., on almost every sample path, the time average on that sample path will be equal to the ensemble average over all paths.
- Thus, assuming ergodicity, we can apply Little's Law in an ensemble-average sense, which we will do.

Proof of Little's Law for Open Systems

- The requirements in Theorem3 are all subsumed (induced, concluded) by the assumption that the system is *ergodic*, (in ergodic systems the above limits all exist)
- Also the average arrival rate and completion rate are equal, since the system empties infinitely often.
- Furthermore, in ergodic systems the time average is equal to the ensemble (or "true") average.
- Thus it is sufficient to require that the system is ergodic for Little's Law, as stated in Theorem1, to hold.

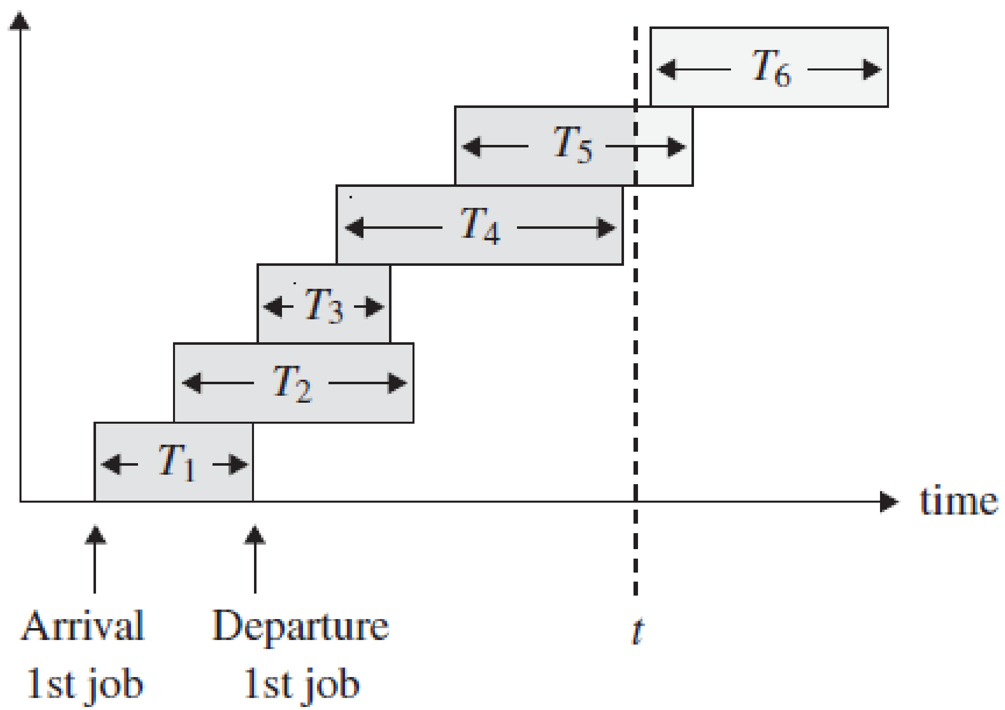
Proof of Little's Law for Open Systems



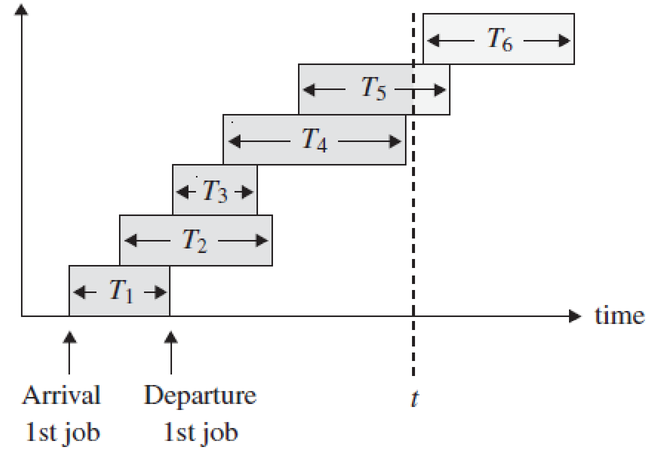
- ***Proof* (*Theorem3*)**
- T_i :the time that the *i*th arrival to the system spends in the system.
- for any time *t*, consider the area, *A*, contained within all the rectangles in Figure 5, up to time *t* (this includes most of the rectangle labeled *T*5).
- 2 views of this area, *A*,
- by summing *horizontally*
- equivalently, by summing *vertically*.

Proof of Little's Law for Open Systems

- **Figure 5.** Graph of arrivals in an open system.
- the area under the curve= *A*,



Proof of Little's Law for Open Systems



the area under the curve= *A*,

A(*t*) = the number of arrivals by time *t*

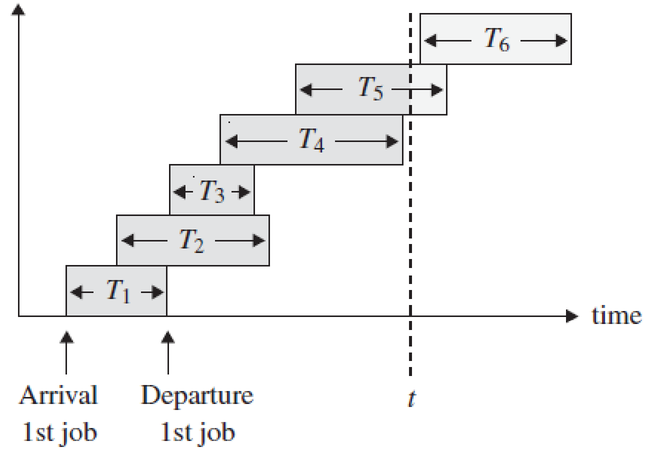
C(*t*) = the number of system completions (departures) by time *t*.

- horizontal view: summing up the T_i's as follows:
- $\sum_{i \in C(t)} T_i \leq A \leq \sum_{i \in A(t)} T_i$

$\sum_{i \in C(t)} T_i$: sum of the time in system of those jobs that have completed by time *t*,

$\sum_{i \in A(t)} T_i$: sum of the time in system of those jobs that have arrived by time *t*.

Proof of Little's Law for Open Systems



- vertical view of *A:*
- adds up the number of jobs in system at any moment in time, $N(s)$, where s ranges from 0 to t =

$$A = \int_0^t N(s)\mathrm{d}s$$

Combining these two views, we have

$$\sum_{i \in C(t)} T_i \leq \int_0^t N(s)\mathrm{d}s \leq \sum_{i \in A(t)} T_i$$

Proof of Little's Law for Open Systems

- Dividing by t throughout, we get

$$\frac{\sum_{i \in C(t)} T_i}{t} \le \frac{\int_0^t N(s) ds}{t} \le \frac{\sum_{i \in A(t)} T_i}{t}$$

- or, equivalently

$$\frac{\sum_{i \in C(t)} T_i}{C(t)} \frac{C(t)}{t} \le \frac{\int_0^t N(s) ds}{t} \le \frac{\sum_{i \in A(t)} T_i}{A(t)} \frac{A(t)}{t}$$

Proof of Little's Law for Open Systems

- Taking limits as $t \to \infty$,

$$\lim_{t\to\infty} \frac{\sum_{i\in C(t)} T_i}{C(t)} \lim_{t\to\infty} \frac{C(t)}{t} \leq \overline{N}^{\text{TimeAvg}}$$

$$\leq \lim_{t\to\infty} \frac{\sum_{i\in A(t)} T_i}{A(t)} \lim_{t\to\infty} \frac{A(t)}{t}$$

$\to \overline{T}^{\text{Time Avg}} . X \leq \overline{N}^{\text{Time Avg}} \leq \overline{T}^{\text{Time Avg}} . \lambda$

we are given that X and λ are equal. Therefore

$\overline{N}^{\text{Time Avg}} = \lambda . \overline{T}^{\text{Time Avg}}$ ▪